Counting Homomorphisms from Hypergraphs of Bounded Generalised Hypertree Width

A Logical Characterisation

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Abstract

→ Investigate the distinguishing power of homomorphisms from hypergraphs of bounded generalised hypertree width.

→ We introduce a simple 2-sorted counting logic GC^k that expresses properties over hypergraphs.

→ We show: Two hypergraphs are distinguishable by the number of homomorphisms from a hypergraph of generalised hypertree width at most k if and only if they can be distinguished by some sentence of the logic GC^k.

Side Result

For hypergraphs G and H it holds that

$$\mathrm{Hom}(\mathsf{GHW}_k, G) = \mathrm{Hom}(\mathsf{GHW}_k, H)$$
$$\Longleftrightarrow$$
$$\mathrm{Hom}(\mathsf{EHW}_k, G) = \mathrm{Hom}(\mathsf{EHW}_k, H)$$

"For homomorphism indistinguishability it is enough to consider the more restrictive entangled hypertree width."

The Logic GC^k

Variables

- Countably many red variables $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \ldots$ representing vertices.
- k blue variables $\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_k$ representing hyperedges.

Atomic formulas

- $E(\mathsf{e}_i, \mathsf{v}_j)$, $\mathsf{v}_i = \mathsf{v}_j$, $\mathsf{e}_i = \mathsf{e}_j$

Negation, Conjunction and Disjunction

- $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$

Guarded Quantification

- $\exists^{\geq n}(\mathsf{v}_{i_1}, \ldots, \mathsf{v}_{i_\ell}).(\Delta_g \wedge \varphi)$
- $\exists^{\geq n}(\mathsf{e}_{i_1}, \ldots, \mathsf{e}_{i_\ell}).(\Delta_g \wedge \varphi)$

where Δ_g has to guard *all* free red variables in φ.

Main Result

For two hypergraphs G and H and any natural number it holds that

$$G \equiv_{\mathsf{GC}^k} H \iff \mathrm{Hom}(\mathsf{GHW}_k, G) = \mathrm{Hom}(\mathsf{GHW}_k, H).$$

"G and H are distinguishable by the number of homomorphisms from a hypergraph F of ghw ≤ k iff they are distinguishable by a sentence of the logic GC^k."

"All hyperedges with ≥ 3 vertices are disjoint."

$$\psi_1 := \exists^{\geq 1}(\mathsf{v}_1).(E(\mathsf{e}_1, \mathsf{v}_1) \wedge E(\mathsf{e}_2, \mathsf{v}_1)) \qquad \psi_2 := \bigwedge_{i \in \{1,2\}} \exists^{\geq 3}(\mathsf{v}_1).(E(\mathsf{e}_i, \mathsf{v}_1))$$

$$\varphi := \neg\exists^{\geq 1}(\mathsf{e}_1, \mathsf{e}_2).(\neg\mathsf{e}_1 = \mathsf{e}_2 \wedge \psi_1 \wedge \psi_2)$$

Proof Overview

$$\mathrm{Hom}(\mathsf{GHW}_k, G) = \mathrm{Hom}(\mathsf{GHW}_k, H)$$
$$\Longleftrightarrow \quad \text{[Böker 2019‡]}$$
$$\mathrm{Hom}(\mathsf{IGHW}_k, I_G) = \mathrm{Hom}(\mathsf{IGHW}_k, I_H)$$
$$\Longleftrightarrow \quad \text{[Thm. 4.1†]}$$
$$\mathrm{Hom}(\mathsf{IEHW}_k, I_G) = \mathrm{Hom}(\mathsf{IEHW}_k, I_H)$$
$$\Longleftrightarrow \quad \text{[Thm. 7.12†]}$$
$$\mathrm{Hom}(\mathsf{GLI}_k, I_G) = \mathrm{Hom}(\mathsf{GLI}_k, I_H)$$
$$\Longleftrightarrow \quad \text{[Thm. 6.1†]}$$
$$G \equiv_{\mathsf{RGC}^k} H$$
$$\Longleftrightarrow \quad \text{[Thm. TBA]}$$
$$G \equiv_{\mathsf{GC}^k} H$$

†: `arXiv:2303.10980v2 [cs.LO]` — use the QR code above.

‡: Böker, J. *Color Refinement, Homomorphisms, and Hypergraphs.* WG 2019.

Hypertree Decompositions

Generalised Hypertree Decomp's

→ Bags (set of vertices) and covers (set of hyperedges) associated with each node of the tree.

→ F.a. vertices v: the bags containing v must form a connected subtree

→ Every bag has to be *covered* by the set of hyperedges.

Entangled Hypertree Decomp's

→ Every EHD is a GHD.

→ F.a. hyperedges e: the bags covered by e must form a connected subtree.

→ Every bag has to be *precisely covered* by the set of hyperedges.

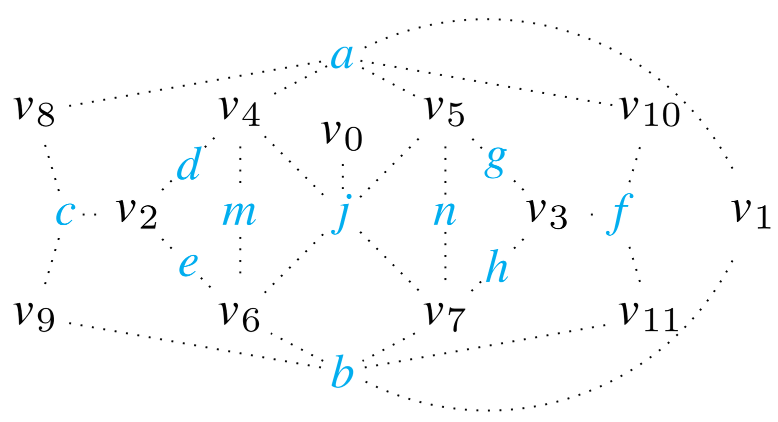
Example of a GHD that is not an EHD

$t_1 : j\{v_0, v_4, v_6, v_5, v_7\}$

$t_2 : a\{v_1, v_4, v_5, v_8, v_{10}\}, b\{v_1, v_6, v_7, v_9, v_{11}\}$

$t_3 : j\{v_4, v_6\}, c\{v_8, v_9, v_2\}$ $t_4 : j\{v_5, v_7\}, f\{v_{10}, v_{11}, v_3\}$

$t_5 : d\{v_4, v_2\}$ $t_6 : e\{v_6, v_2\}$ $t_7 : g\{v_5, v_3\}$ $t_8 : h\{v_7, v_3\}$



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