Counting Homomorphisms from Hypergraphs of Bounded Generalised Hypertree Width

A Logical Characterisation

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What is this about?

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Slides available at `hu.berlin/adhoc23`
`arXiv:2303.10980`

Motivation and Main Result

GC^k — A 2-sorted Counting Logic with Guards

About the Proof

Conclusion

Homomorphism Indistinguishability

We saw today that there are many results on graphs e.g.:

- Lovász 1967
- Dvořák 2010; Dell, Grohe, and Rattan 2018
- Mančinska and Roberson 2020

What about *hyper*graphs?

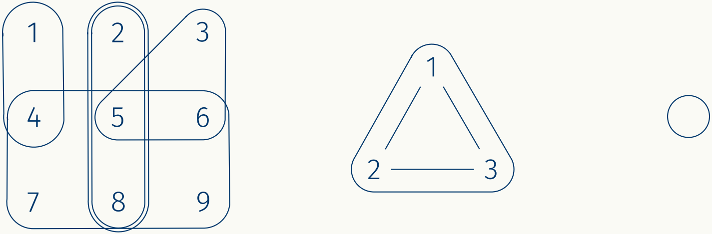
- Böker 2019:

$$\mathrm{Hom}_{\mathrm{Berge}}(H_1) = \mathrm{Hom}_{\mathrm{Berge}}(H_2) \quad \iff \quad I_{H_1} \equiv_{\text{1-WL}} I_{H_2}$$

Can we do more?

What are Hypergraphs?

A hypergraph H is a finite set $V(H)$ of vertices and a finite **multi**set $E(H)$ of edges $e \subseteq V(H)$.



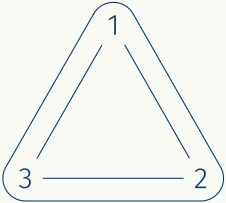
Incidence Graphs

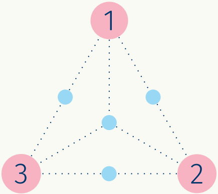
Definition

The incidence graph $I_H = (R, B, E)$ of H is defined as:

$$R = V(H) \quad B = E(H) \quad ({\color{cyan}e}, {\color{red}v}) \in E \iff v \in f_H(e)$$

i.e. draw an edge if v is contained in the hyperedge e.





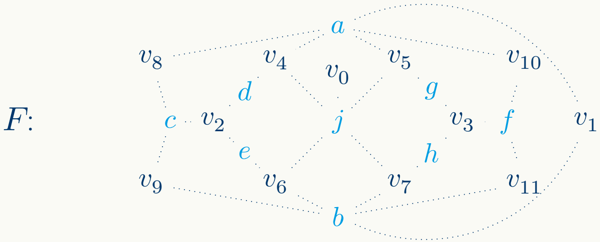
Generalised Hypertree Width

- Generalisation of treewidth to hypergraphs.
- Based on (Hyper)tree decompositions – as usual:
  - Bags of vertices associated with each node of the tree
  - F.a. vertices v: the bags containing v must form a connected subtree.

The twist:

- Every bag has to be *covered* by a set of hyperedges
  - I.e. the union of the hyperedges must be a superset of the bag.
- The width of a decomposition is the size of the biggest cover.

Example



$t_1 : j\{v_0, v_4, v_6, v_5, v_7\}$

$t_2 : a\{v_1, v_4, v_5, v_8, v_{10}\}, b\{v_1, v_6, v_7, v_9, v_{11}\}$

$t_3 : j\{v_4, v_6\}, c\{v_8, v_9, v_2\}$ $\quad$ $t_4 : j\{v_5, v_7\}, f\{v_{10}, v_{11}, v_3\}$

$t_5 : d\{v_4, v_2\}$ $\quad$ $t_6 : e\{v_6, v_2\}$ $\quad$ $t_7 : g\{v_5, v_3\}$ $\quad$ $t_8 : h\{v_7, v_3\}$

Main Result – Counting Hyper-what?!

Theorem

Let G, H be hypergraphs, $k \in \mathbb{N}_{\geqslant 1}$.

$$\mathsf{Hom}_{\mathsf{GHW}_k}(G) = \mathsf{Hom}_{\mathsf{GHW}_k}(H) \iff G \equiv_{\mathsf{GC}^k} H$$

Theorem (Dvořák 2010; Dell, Grohe, and Rattan 2018)

Let G, H be graphs, $k \in \mathbb{N}_{\geqslant 1}$.

$$\mathsf{Hom}_{\mathsf{TW}_k}(G) = \mathsf{Hom}_{\mathsf{TW}_k}(H) \iff G \equiv_{\mathsf{C}^{k+1}} H$$

GC^k is a new logic introduced by us.

Introducing GC^k — The Basics

- Blue and red Variables

$$\begin{aligned} \mathsf{VAR} := \{\mathsf{e}_1, \dots, \mathsf{e}_k\} \quad & \text{represent edges} \\ \cup \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \dots\} \quad & \text{represent vertices} \end{aligned}$$

- Guard function $g : \mathbb{N}_{\geqslant 1} \rightharpoonup [k]$

$$\Delta_g := \bigwedge_{i \in \mathsf{dom}(g)} E(\mathsf{e}_{g(i)}, \mathsf{v}_i) \qquad \text{“}v_i \in e_{g(i)}\text{”}$$

$$\Delta_g := \top \text{ for } \mathsf{dom}(g) = \emptyset$$

- red variables are always guarded by some blue variable

Introducing GC^k — Definition

We have...

Atomic formulas:

- $E(\mathsf{e}_i, \mathsf{v}_j), \quad \mathsf{v}_i = \mathsf{v}_j, \quad \mathsf{e}_i = \mathsf{e}_j$

Negation, Conjunction and Disjunction:

- $\neg\varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi)$

Guarded Quantification:

- $\exists^{\geq n}(\mathsf{v}_{i_1}, \dots, \mathsf{v}_{i_\ell}).(\Delta_g \wedge \varphi)$
- $\exists^{\geq n}(\mathsf{e}_{i_1}, \dots, \mathsf{e}_{i_\ell}).(\Delta_g \wedge \varphi)$

where Δ_g has to guard *all* free red variables in φ.

GC^k – Example

Find φ that expresses:

> *All hyperedges containing at least 3 vertices are pairwise disjoint.*

$$\varphi := \neg\exists^{\geq 1}(\mathsf{e}_1, \mathsf{e}_2)\,.\,(\underbrace{\top}_{\text{Guard}} \wedge (\neg \mathsf{e}_1 = \mathsf{e}_2 \wedge \psi_{\text{not disjoint}} \wedge \psi_{\geqslant 3\text{ vertices}}))$$

with

$$\psi_{\text{not disjoint}} := \exists^{\geq 1}(\mathsf{v}_1)\,.\,(\underbrace{E(\mathsf{e}_1, \mathsf{v}_1)}_{\text{Guard}} \wedge \bigwedge_{j \in \{1,2\}} E(\mathsf{e}_j, \mathsf{v}_1))$$

$$\psi_{\geqslant 3\text{ vertices}} := \bigwedge_{j \in \{1,2\}} \exists^{\geq 3}(\mathsf{v}_1)\,.\,(\underbrace{E(\mathsf{e}_j, \mathsf{v}_1)}_{\text{Guard}} \wedge E(\mathsf{e}_j, \mathsf{v}_1))$$

Proof 🧐

...sort of

Caution when dealing with homomorphisms

Definition (Homomorphisms on Hypergraphs)

Let $h_V : V(H) \to V(G)$ and $h_E : E(H) \to E(G)$.
(h_V, h_E) is a homomorphism from G to H if for all $e \in E(G)$:

$$f_H(h_E(e)) = \{\, h_V(v) \mid v \in f_G(e) \,\}$$

$$\not\equiv$$

Böker 2019 implicitly shows that this is not problematic

Definition (Homomorphisms on Incidence Graphs)

Let $h_R : R(I_G) \to R(I_H)$ and $h_B : B(I_G) \to B(I_H)$.
(h_R, h_B) is a homomorphism from I_G to I_H if:

$$(e, v) \in E(I_G) \implies (h_B(e), h_R(v)) \in E(I_H)$$

About that Proof...

Step 1: Relate homomorphisms over hypergraphs and incidence graphs

- **Recall**: Not the same — but Böker 2019 solved this already

Step 2: Relate ghw to restricted variant ehw

- Gives us an inductive characterisation

Step 3: Find suitable “normal form” NGC^k for GC^k

Step 4: Relate ehw and NGC^k

- Adapt beautiful machinery in spirit of Courcelle 1993 to incidence graphs
- **Nasty** — Like getting two cats to the vet using a single pet carrier 🐈📦🐈‍⬛
  - Stuff one cat in the box and the other escapes — Repeat until they accept their fate

Step 5: Profit 🎉

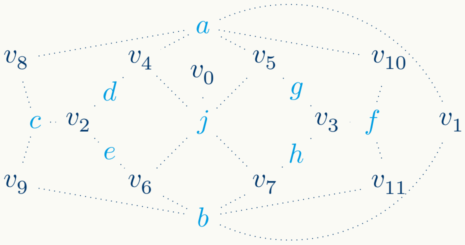
Generalised Hypertree Width

A generalised hypertree decomposition is a tree with

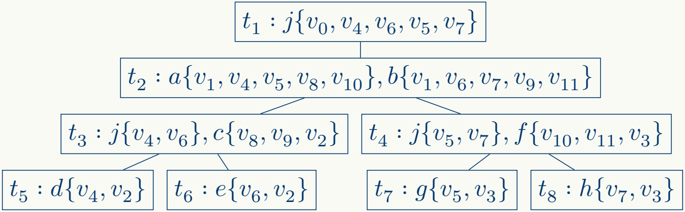
- Bags (set of vertices) and covers (set of hyperedges) associated with each node of the tree
- F.a. vertices v: the bags containing v must form a connected subtree
- F.a. hyperedges e: the bags covered by e must form a connected subtree
- Every bag has to be precisely *covered* by a set of hyperedges
  - I.e. the union of the hyperedges must be equal to the bag
- The width of a decomposition is the size of the biggest cover.

Example

F:



a
v_8
v_4
v_0
v_5
v_{10}
d
g
c
v_2
j
v_3
f
v_1
e
h
v_9
v_6
v_7
v_{11}
b



$t_1 : j\{v_0, v_4, v_6, v_5, v_7\}$
$t_2 : a\{v_1, v_4, v_5, v_8, v_{10}\}, b\{v_1, v_6, v_7, v_9, v_{11}\}$
$t_3 : j\{v_4, v_6\}, c\{v_8, v_9, v_2\}$
$t_4 : j\{v_5, v_7\}, f\{v_{10}, v_{11}, v_3\}$
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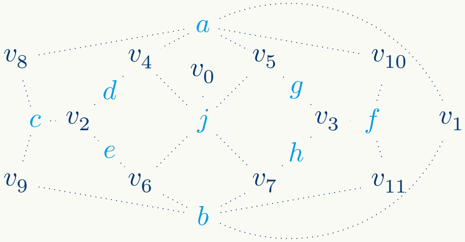
Relating ghw and ehw

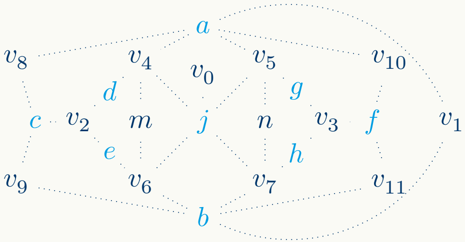
F:



- Assume we want to use F with $\mathrm{ghw}(F) = 2$ to distinguish some G and H by number of homomorphisms.
- But $\mathrm{ehw}(F) = 3$.
- Fix: Turn F into F' with $\mathrm{ehw}(F') = 2$ while keeping distinguishing property.

Simple Idea: Just add more edges 🤷‍♂️

F':



$t_1 : j\{v_0, v_4, v_6, v_5, v_7\}\}$

$t_2 : a\{v_1, v_4, v_5, v_8, v_{10}\}, b\{v_1, v_6, v_7, v_9, v_{11}\}$

$t_3 : m\{v_4, v_6\}, c\{v_8, v_9, v_2\}$

$t_4 : n\{v_5, v_7\}, f\{v_{10}, v_{11}, v_3\}$

$t_5 : d\{v_4, v_2\}$

$t_6 : e\{v_6, v_2\}$

$t_7 : g\{v_5, v_3\}$

$t_8 : h\{v_7, v_3\}$

But this changes the number of homomorphisms!

Relating ghw and ehw

Lemma (simplified)

Let G, H, F be hypergraphs s.t. $\hom(F, G) \neq \hom(F, H)$ *and let* $e \in E(F)$.

Then, for every $s \subseteq e$ *and every* $x \in \mathbb{N}$ *there exists a* $y \geq x$ *such that* $\hom(F + y \cdot s, G) \neq \hom(F + y \cdot s, H)$.

Simpler:

> *"If we want to add x copies of a subset of some hyperedge to F, we can retain its distinguishing property as long as we can handle some additional copies."*

I.e. insert copies of (sub)edges to enforce precise coverage and connectedness for edges.

Concluding Remarks

Theorem

The following statements are equivalent:

1. *Ex.* $\varphi \in \mathsf{GC}^k$ *s.t.* $G \vDash \varphi$ *and* $H \nvDash \varphi$
2. *Ex.* F *with* $\mathsf{ghw}(F) \leq k$ *s.t.* $\mathsf{hom}(F, G) \neq \mathsf{hom}(F, H)$

So...

- ...how hard is it to compute the homomorphism vector?
- ...EF-games for GC^k?
- ...does Grohe 2020a generalise to hypertree-depth?
- ...Weisfeiler-Leman on hypergraphs? Ongoing work!

Related Work i

📝 Böker, Jan (2019). “Color Refinement, Homomorphisms, and Hypergraphs”.
DOI: 10.1007/978-3-030-30786-8_26.

📝 Cai, Jin-Yi, Martin Fürer, and Neil Immerman (1992). “An optimal lower bound on the number of variables for graph identification”.
DOI: 10.1007/BF01305232.

📝 Courcelle, Bruno (1993). “Graph Grammars, Monadic Second-Order Logic And The Theory Of Graph Minors”.
DOI: 10.1090/conm/147.

Related Work ii

📝 Dell, Holger, Martin Grohe, and Gaurav Rattan (2018). "Lovász Meets Weisfeiler and Leman".
DOI: 10.4230/LIPIcs.ICALP.2018.40.

📝 Dvořák, Zdeněk (2010). "On recognizing graphs by numbers of homomorphisms".
DOI: 10.1002/jgt.20461.

📝 Gottlob, Georg, Nicola Leone, and Francesco Scarcello (2002). "Hypertree Decompositions and Tractable Queries".
DOI: 10.1006/jcss.2001.1809.

📝 Grohe, Martin (2020a). "Counting Bounded Tree Depth Homomorphisms".
DOI: 10.1145/3373718.3394739.

Related Work iii

- 📝 Grohe, Martin (2020b). "Word2vec, Node2vec, Graph2vec, X2vec: Towards a Theory of Vector Embeddings of Structured Data".
  DOI: 10.1145/3375395.3387641.
- 📝 Grohe, Martin (2021). "The Logic of Graph Neural Networks".
  DOI: 10.1109/LICS52264.2021.9470677.
- 📝 Lovász, László (1967). "Operations with structures".
  DOI: 10.1007/BF02280291.
- 📝 Mančinska, Laura and David E. Roberson (2020). "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs".
  DOI: 10.1109/FOCS46700.2020.00067.

Definition (Gottlob, Leone, and Scarcello 2002)

A complete generalised hypertree decomposition $D := (T, cover, bag)$ of H is a tree T with functions $cover : V(T) \to E(H)$ and $bag : V(T) \to V(H)$ s.t.

1. F.a. $e \in E(H)$ ex. $n \in V(T)$ s.t. $e \in cover(n)$ and $f(e) \subseteq bag(n)$
2. F.a. $v \in V(H)$ the subgraph T_v induced by $V_v := \{n \in V(T) \mid v \in bag(n)\}$ is a tree.
3. F.a. $n \in V(T)$ we have $bag(n) \subseteq \bigcup_{e \in cover(n)} f(e)$.

On Hypertree Decompositions

Definition

A complete entangled hypertree decomposition $D := (T, cover, bag)$ of H is a tree T with functions $cover : V(T) \to E(H)$ and $bag : V(T) \to V(H)$ s.t.

1. F.a. $e \in E(H)$ ex. $n \in V(T)$ s.t. $e \in cover(n)$ and $f(e) \subseteq bag(n)$
2. F.a. $v \in V(H)$ the subgraph T_v induced by $V_v := \{n \in V(T) \mid v \in bag(n)\}$ is a tree.
3. F.a. $n \in V(T)$ we have $bag(n) = \bigcup_{e \in cover(n)} f(e)$.
4. F.a. $e \in E(H)$ the subgraph T_e induced by $V_e := \{n \in V(T) \mid e \in cover(n)\}$ is a tree.